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BUCKLING OF THIN CIRCULAR RINGS UNDER UNIFORM PRESSURE

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Abstract—The twist-buckling of thin rings is investigated. A simple formula for calculating the in-plane buckling of the ring with extensional effects included, is also derived. The load required for twist-buckling is found to be generally less than that for in-plane buckling.

NOTATION

A	cross-sectional area of ring
a	mean radius
E	elastic tensile modulus
G	shear modulus
I	moment of inertia of ring section about axis normal to plane of ring
$\bar{I}$	moment of inertia about radial axis
I_p	polar moment of inertia
J	torsion constant
n	integer
p	radial (inward) pressure
t	time
u	displacement normal to plane of ring
v	tangential displacement
w	radial displacement
α	$GJ/E\bar{I}$
β	I_p/Aa^2
θ	rotation of ring
κ	Aa^2/I
λ	$pa^3/E\bar{I}$
μ	pa^3/EI
ρ	mass density
ϕ	circumferential coordinate

INTRODUCTION

THE problem of the buckling of a thin ring in its plane was solved nearly a century ago [1, 2]. The solution assumes the inextensibility of the centerline, an entirely reasonable assumption which has stood the test of time. It is well known, however, that in many applications a circular ring does not fail by buckling in its plane—unless some external constraint is imposed to cause it to so buckle—but, rather, twists and buckles out of its plane. It is indeed surprising that this very practical problem has not received the attention that it deserves. Apparently, the only study of this subject is that by Goldberg and Bogdanoff [3] who, however, confined attention to a ring of I section.

As a secondary result of our study we derive a simple equation for calculating the in-plane buckling of rings of arbitrary section with extensional effects included. While the effects of extensionality have been treated previously [4, 5], no simple formula for extensional buckling, comparable to the classical one for inextensional buckling, apparently exists in the literature.

It turns out that it is a tedious and difficult procedure to derive the differential equations governing the torsional buckling of a ring of arbitrary section from first principles. We have overcome this difficulty by first stating the equations of free vibration of circular rings for all types of motion, which are available in the literature, and then using *ad hoc* arguments for deducing therefrom the corresponding equations governing the buckling of a ring.

The equations for all types of vibration of a circular ring have been derived by Love [6], and the equations for in-plane vibrations also by Lamb [7]. Their equations differ in minor respects because Lamb does not assume the inextensibility of the centerline and uses a slightly different term for the in-plane curvature.

Our equations for in-plane vibrations differ slightly from those of Lamb, being symmetrical, whereas those of Lamb are unsymmetrical. By using the expression for in-plane curvature given by Love, while retaining the expression for the extension of the centerline as Lamb does, a symmetrical set of equations emerges.

DIFFERENTIAL EQUATIONS

We shall not elaborate upon the details of the derivation of the equations for the free vibrations of circular rings because they can be deduced in a fairly straightforward manner from Lamb's and Love's equations. The equations are:

$$\frac{EI}{a^4}\left(\frac{\partial^4 w}{\partial\phi^4}-\frac{\partial^3 v}{\partial\phi^3}\right)+\frac{EA}{a^2}\left(w+\frac{\partial v}{\partial\phi}\right)+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

$$\frac{EI}{a^4}\left(\frac{\partial^3 w}{\partial\phi^3}-\frac{\partial^2 v}{\partial\phi^2}\right)-\frac{EA}{a^2}\left(\frac{\partial w}{\partial\phi}+\frac{\partial^2 v}{\partial\phi^2}\right)+\rho A\frac{\partial^2 v}{\partial t^2}=0 \tag{2}$$

$$\frac{E\bar{I}}{a^4}\left(\frac{\partial^4 u}{\partial\phi^4}-\frac{\partial^2(a\theta)}{\partial\phi^2}\right)-\frac{GJ}{a^4}\left(\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2(a\theta)}{\partial\phi^2}\right)+\rho A\frac{\partial^2 u}{\partial t^2}=0 \tag{3}$$

$$\frac{E\bar{I}}{a^2}\left(a\theta-\frac{\partial^2 u}{\partial\phi^2}\right)-\frac{GJ}{a^2}\left(\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2(a\theta)}{\partial\phi^2}\right)+\rho I_p\frac{\partial^2(a\theta)}{\partial t^2}=0 \tag{4}$$

where w is the radial displacement positive outward, v the tangential displacement, u the displacement at right angles to the plane of ring and θ the angular rotation of a cross section. A is the cross-sectional area of the ring of radius a, I and $\bar{I}$ are the principal moments of inertia of the cross section about axes at right angles to the plane of the ring and a radial axis, respectively, $I_p = I+\bar{I}$ is the polar moment of inertia, J the torsion constant of the cross section and ρ the mass density. E and G are the elastic tensile modulus and shear modulus, respectively, and t is time. ϕ is the circumferential coordinate.

As may be expected, the radial and tangential motions are coupled, being governed by equations (1) and (2). Similarly, the out-of-plane and torsional motions are coupled and governed by equations (3) and (4). These two types of motion can be investigated separately.

TRANSITION TO BUCKLING EQUATIONS

We consider a circular ring under a uniform inward radial pressure p per unit of circumferential length. The compressive force in the ring is pa, where a is the radius. This compressive force may cause buckling of the ring either in its plane or out of its plane.

In deducing the corresponding buckling equations from equations (1) to (4), we use a device which has been suggested by Timoshenko in investigating the torsional buckling of open section columns. His procedure is merely to replace the external load term by a fictitious load whose intensity is the load causing buckling (here pa) times the appropriate "curvature" term.

On this basis, we may deduce the buckling equations from the equations of vibration by formally replacing the inertia terms as follows:

$$\begin{aligned} \rho &\rightarrow pa/A \\ \frac{\partial^2 w}{\partial t^2} &\rightarrow \frac{1}{a^2}\frac{d}{d\phi}\left(\frac{dw}{d\phi}-v\right) \\ \frac{\partial^2 v}{\partial t^2} &\rightarrow \frac{1}{a^2}\frac{d}{d\phi}\left(w+\frac{dv}{d\phi}\right) \\ \frac{\partial^2 u}{\partial t^2} &\rightarrow \frac{1}{a^2}\frac{d}{d\phi}\left(\frac{du}{d\phi}\right) \\ \frac{\partial^2 (a\theta)}{\partial t^2} &\rightarrow \frac{1}{a^2}\frac{d}{d\phi}\left(\frac{da\theta}{d\phi}\right) \end{aligned} \tag{5}$$

It may be noted that $(1/a)(dw/d\phi - v)$ is the in-plane slope, $(1/a)(w+dv/d\phi)$ is the circumferential strain, $(1/a)(du/d\phi)$ is the slope at right angles to the plane of the ring and $(1/a)d(a\theta)/d\phi$ is the rate of change of angle of twist. We thus obtain the following set of equations governing the buckling of a circular ring.*

$$\frac{EI}{a^4}\left(\frac{d^4 w}{d\phi^4}-\frac{d^3 v}{d\phi^3}\right)+\frac{EA}{a^2}\left(w+\frac{dv}{d\phi}\right)+\frac{p}{a}\left(\frac{d^2 w}{d\phi^2}-\frac{dv}{d\phi}\right) = 0 \tag{6}$$

$$\frac{EI}{a^4}\left(\frac{d^3 w}{d\phi^3}-\frac{d^2 v}{d\phi^2}\right)-\frac{EA}{a^2}\left(\frac{dw}{d\phi}+\frac{d^2 v}{d\phi^2}\right)+\frac{p}{a}\left(\frac{dw}{d\phi}+\frac{d^2 v}{d\phi^2}\right) = 0 \tag{7}$$

$$\frac{E\bar{I}}{a^4}\left[\frac{d^4 u}{d\phi^4}-\frac{d^2(a\theta)}{d\phi^2}\right]-\frac{GJ}{a^4}\left[\frac{d^2 u}{d\phi^2}+\frac{d^2(a\theta)}{d\phi^2}\right]+\frac{p}{a}\frac{d^2 u}{d\phi^2} = 0 \tag{8}$$

$$\frac{E\bar{I}}{a^2}\left(a\theta-\frac{d^2 u}{d\phi^2}\right)-\frac{GJ}{a^2}\left[\frac{d^2 u}{d\phi^2}+\frac{d^2(a\theta)}{d\phi^2}\right]+\frac{p}{A}\frac{I_p}{a}\frac{d^2(a\theta)}{d\phi^2} = 0 \tag{9}$$

Since equations (1) through (4) are valid for both full and partial rings which can bend in their plane without twisting, so also are the equations (6) through (9). These equations may therefore be used to investigate the buckling of circular arches.

* It is implied that the displacements are to be measured from the position of stable equilibrium defined by $w_0 = -pa^2/AE, v = u = \theta = 0$.

IN-PLANE BUCKLING

Equations (6) and (7) containing both the displacement components w and v are capable of satisfying any suitable set of boundary conditions for both complete and incomplete rings. They are therefore more general than the customary formulations in which the condition of inextensibility $w+\mathrm{d}v/\mathrm{d}\phi = 0$ is assumed *ab initio*.

Let

$$\begin{aligned} w &= W\sin(n\phi+\phi_0) \\ v &= V\cos(n\phi+\phi_0) \end{aligned} \tag{10}$$

where W, V and ϕ_0 are constants.

Substitution in equations (6) and (7) results in the two equations

$$\begin{aligned} (n^4+\kappa-\mu n^2)W - n(n^2+\kappa-\mu)V &= 0 \\ -n(n^2+\kappa-\mu)W + n^2(1+\kappa-\mu)V &= 0 \end{aligned} \tag{11}$$

with

$$\kappa = Aa^2/I, \qquad \mu = pa^3/EI.$$

The condition for the vanishing of the determinant of the coefficients in equations (11) gives

$$\mu^2 - \mu(\kappa+n^2) + \kappa(n^2-1) = 0, \qquad n \neq 0, 1. \tag{12}$$

Equation (12) has two positive roots for any integer $n \geq 2$, the smaller of which gives the buckling load. ($n = 0, 1$ represent rigid body displacements of the ring.)

We note that κ is a large quantity in a thin ring, and the sum of the roots, $\kappa+n^2$, must be nearly equal to κ. The product of the roots is $\kappa(n^2-1)$ so that the larger root must be very nearly equal to κ. Thus, the smaller root is, very nearly,

$$\mu_{cr} \approx \kappa(n^2-1)/\kappa = (n^2-1)$$

so

$$p_{cr} \approx (n^2-1)EI/a^3, \qquad n = 2, 3 \ldots \tag{13}$$

a familiar result. $n = 2$, of course, gives the buckling load. When $\mu = n^2-1$, it follows from equations (11) that $W = nV$ with κ large. Thus, the centerline is essentially inextensible for thin rings. Figure 1 gives the buckling load for various values of κ as computed from equation (12). It may be noted that $\mu \to 3$, the classical value, as $\kappa \to \infty$. Thus, the thinner the ring, the closer the buckling load is to the classical value. That the inclusion of the extension would decrease the buckling load could, of course, have been anticipated from Rayleigh's principle, and it seems physically reasonable that extensional effects should be more pronounced in thicker rings.

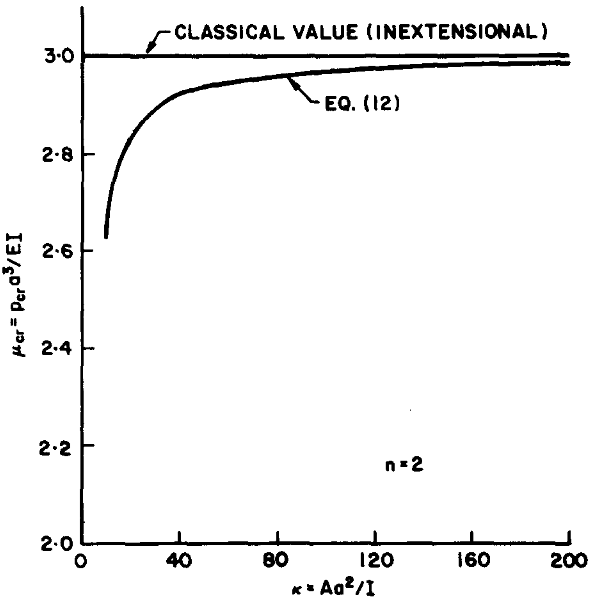

FIG. 1. In-plane buckling of rings.

TWIST BUCKLING

Let

$$\begin{aligned} u &= U \sin(n\phi + \phi_0) \\ a\theta &= \Theta \sin(n\phi + \phi_0) \end{aligned} \tag{14}$$

where U, Θ and ϕ_0 are constants and n is an integer. Introduction into equations (8) and (9) gives

$$\begin{aligned} (n^4 + \alpha n^2 - \lambda n^2)U + n^2(\alpha + 1)\Theta &= 0 \\ n^2(\alpha + 1)U + (1 + \alpha n^2 - \beta\lambda n^2)\Theta &= 0 \end{aligned} \tag{15}$$

with

$$\alpha = GJ/E\bar{I}, \qquad \lambda = pa^3/E\bar{I}, \qquad \beta = I_p/Aa^2$$

The determinantal equation takes the form

$$(n^2 + \alpha - \lambda)(1 + \alpha n^2 - \beta\lambda n^2) - n^2(1 + \alpha)^2 = 0, \qquad n \neq 0$$

which may also be written

$$\lambda^2 - \lambda\left(n^2 + \alpha + \frac{\alpha}{\beta} + \frac{1}{\beta n^2}\right) + \frac{\alpha(n^2 - 1)^2}{\beta n^2} = 0 \tag{16}$$

This quadratic gives two positive roots for all $n \geq 2$, the smallest of which is the root sought. It may be shown that this root occurs for $n = 2$. Some solutions of equation (16) for various type sections are given in Fig. 2.

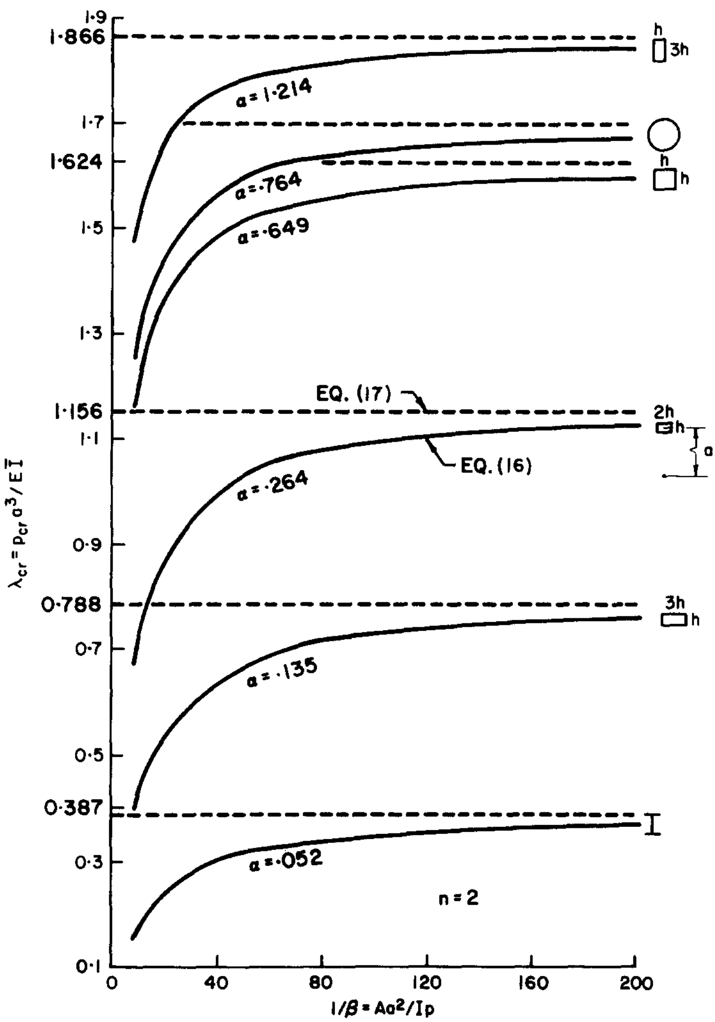

FIG. 2. Twist buckling of rings.

When $n = 1$, equation (16) gives only one root, viz:

$$\lambda = (1+\alpha)\left(1+\frac{1}{\beta}\right) \approx \frac{(1+\alpha)}{\beta}, \qquad \text{since } \beta \ll 1$$

This root is a large one and is inadmissible for calculating p_{cr}. It represents a rigid body rotation of the ring about a diameter followed by a uniform twist, in opposite directions, of the two halves.

As β becomes very small, it may be shown that for the usual values of α, the smaller root is given very nearly by

$$\begin{aligned} \lambda_{cr} &\approx \frac{\alpha(n^2-1)^2}{\beta n^2} \bigg/ \frac{(\alpha n^2+1)}{\beta n^2}, \qquad n = 2 \\ &\approx 9\alpha/(1+4\alpha) \end{aligned} \tag{17}$$

and so

$$p_{cr} \approx \frac{9\alpha}{1+4\alpha}\frac{E\bar{I}}{a^3} \tag{18}$$

Equation (17) is shown by dashed lines in Fig. 2.

It may be noted from Fig. 2 that the load required for twist-buckling is smaller than that for in-plane buckling for all sections shown except the "wide" section which has a large value of $\bar{I}$. It would seem therefore that twist-buckling rather than in-plane buckling will be the controlling criterion for most sections unless special conditions exist to prevent the ring from twisting out of its plane.

As noted in the Introduction, Goldberg and Bogdanoff [3] deduced an equation for the out-of-plane buckling of circular rings of I section. They assumed in their derivation that the plane of action of the radial load, which was supposed acting at the inner flange, moved parallel to itself as the section rotated. In our solution, equation (16), the line of action of the pressure p remains unchanged as the section rotates. If one omits the term Rb_p/n^2 in their equation (40), the conditions assumed in the two solutions become identical.

If the solution of Ref. [3] is changed as noted above, the resulting equation is still considerably more complex than equation (16) of this paper because of the presence in the former of many small terms. It is nevertheless instructive to compare the results obtained by using the two solutions.

We take for illustration the numerical example given by Goldberg and Bogdanoff. The relevant quantities are

$A = 19{\cdot}4\ \text{in}^2$ $\qquad a = 727{\cdot}7$ in.

$I_p = 886{\cdot}15\ \text{in}^4$ $\qquad E = 30\times 10^6\ \text{lb/in}^2$

$\bar{I} = 106{\cdot}28\ \text{in}^4$ $\qquad G = 11{\cdot}5\times 10^6\ \text{lb/in}^2$

$J = 2{\cdot}352\ \text{in}^4$

The modified equation of Ref. [3] then yields the buckling load

$$N_{cr} = (pa)_{cr} = 453{\cdot}9\ \text{lb}, \qquad n = 2$$

Equation (16) of this paper yields

$$N_{cr} = (pa)_{cr} = 444\ \text{lb}, \qquad n = 2$$

and the approximate relation, equation (18), yields

$$N_{cr} = 444{\cdot}6\ \text{lb}$$

The results are clearly very close. It is worthy of note that when equation (40) of Ref. [3] is modified as indicated above it yields positive roots only, so that buckling with a tensile hoop tension which was found with a translating load vector is no longer possible. Furthermore, the assumption that the load vector does not translate always gives the smaller critical load.

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Résumé—Le fléchissement par torsion d'anneaux minces est examiné. Une formule simple pour le calcul du fléchissement dans le plan de l'anneau y compris des effets d'allongement, est aussi dérivée. La charge nécessaire pour le fléchissement par torsion apparait être généralement inférieure à celle pour le fléchissement dans le plan.

Zusammenfassung—Das Drillknicken dünner Ringe wird untersucht, Eine einfache Formel wird abgeleitet die das Errechnen des Knickens in der Ebene, mit Streckungswirkungen, ermöglichen. Die Last die notwendig ist um Drillknicken zu bewirken ist im allgemeinen geringer als für die ebene Kinckung.

Абстракт—Исследуется выпучивание с кручеником тонких колец. Выведена также формула для расчета выпучивания кольца в плоскости, при наличии добавочных эффектов. Найденная нагрузка для выпучивания с кручением является вобщем менее чем для выпучивания в плоскости.